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(Residential Autonomous College under University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2014

SECOND YEAR

MATHEMATICS (Honours)

Date : 23/05/2014

Time : 11 am – 3 pm

Paper : IV

Full Marks : 100

[Use a Separate Answer Book for each group]

Group – A

Unit – I

(*Answer **any five** questions*)

1. a) In $\mathbb{R}$ with usual metric, find two disjoint uncountable closed sets A, B such that d(A,B) = 0.
   b) In $\mathbb{R}$ with usual metric, find four distinct subsets A,B,C,D such that BdA=BdB=BdC=BdD={0,1}.
   c) If (X,d) is a metric space, $A \subseteq X$; prove that BdA = Bd(X–A). [2+2+3]

2. a) If X is an uncountable subset of $\mathbb{R}$, show that X has a limit point and also show that there exists a subset Y of X which is not closed in $\mathbb{R}$.
   b) Give example of subsets $A, B \subseteq Q$ such that $d(A,B) = \sqrt{2}$. [(3+2)+2]

3. Let $X = \{\{x_n\} \mid x_n \in \mathbb{R}\ \forall n \ \& \sup_n |x_n| < \infty\}$

   For $x = \{x_n\} \ \& \ y = \{y_n\}$ in X define $d(x,y) = \sup_n |x_n - y_n|$.
   a) Prove that (X,d) is a metric space.
   b) If $S = \{\{x_n\} \in X \mid x_n = 0 \text{ or } 1 \ \forall n\}$ then show that $x \neq y$ in S implies $d(x,y) \geq 1$.
   c) Use (b) and prove that (X,d) is not 2nd countable. [3+2+2]

4. a) Let $\{U_n\}$ be a sequence of dense open subsets in a complete metric space (X,d). Prove that $\bigcap_{n=1}^{\infty} U_n$ is also dense in X.
   b) Use (a) and prove that the set of rationals is not a G_δ subset of $\mathbb{R}$. [4+3]

5. a) Let (X,d) be a metric space and A,B be two disjoint closed sets in X. Prove that there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0 \ \forall x \in A$ and $f(x) = 1 \ \forall x \in B$.

      Use above result to show that a metric space is normal.
   b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous maps and $f|_{\mathbb{Q}} = g|_{\mathbb{Q}}$ show that f = g. [(3+2)+2]

6. a) Let A be a nonempty subset of $\mathbb{R}^n$ with the Euclidean metric. Define $f : \mathbb{R}^n \to \mathbb{R}$ by f(x)=d(x,A). Assume that f is a continuous function.

      Fix $x_0 \in \mathbb{R}^n$. Show that if A is a compact subset of $\mathbb{R}^n$ then $\exists a \in A$ such that $d(x_0,a) = d(x_0,A)$.
   b) If A is closed instead of compact does there exists $a \in A$ such that $d(x_0,a) = d(x_0,A)$? Justify your answer. [4+3]

7. a) Show that a metric space (X,d) is totally bounded if and only if every sequence in X has a Cauchy subsequence.
   b) Use (a) to conclude that every bounded sequence of real umbers has a convergent subsequence. [5+2]

8. a) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(\mathbb{Q}) \subseteq \mathbb{R} - \mathbb{Q}$ and $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$. Show that f is not continuous.
   b) Show that $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is a path connected subset of $\mathbb{R}^3$ with Euclidean metric. [3+4]

Unit – II

(*Answer* **any three** *questions*)

9. Let $D \subseteq \mathbb{R}$ and let $\{f_n\}$ be sequence of functions pointwise convergent on D to a function f. Let $M_n = \sup_{x \in D} |f_n(x) - f(x)|$. Show that $\{f_n\}$ is uniformly convergent on D if and only if $\lim_{n\to\infty} M_n = 0$. Hence show that $f_n = 1 - \frac{x^n}{n}, x \in [0,1]$ is uniformly convergent. [3+2]

10. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each $n \in \mathbb{N}$, define $f_n(x) = f\left(x + \frac{1}{n}\right)\ \forall x \in \mathbb{R}$. Prove that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$. [5]

11. Let $\sum_{n=1}^{\infty} f_n$ be a series of function converging uniformly to the function f on [a,b]. Then
    a) If f_n are all continuous then is f continuous? Justify your answer.
    b) If f_n are all differentiable then is f differentiable? Justify your answer. [3+2]

12. a) Find the radius of convergence of the power series $x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \ldots + \frac{(n!)^2}{(2n)!}x^n + \ldots$
    b) Let $\sum a_n x^n$ be a power series with radius of convergence R(>0). Construct a power series $\sum b_n x^n$ other than $\sum a_n x^n$ such that the radius of convergence of the series $\sum a_n b_n x^n$ is also R. [2+3]

13. a) Find the radius of convergence of $\sum_{n=1}^{\infty}\left(\frac{1}{3^n} + \frac{1}{5^n}\right)x^n$.
    b) Find the radius of convergence of $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$. Use Abel's theorem and find the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$. [3+2]

Group-B

Unit-I

Answer **any three** questions: 3x10=30

14. (a) Solve: $\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$ 4
    (b) Find the partial differential equation arising from $\phi\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$, where ϕ is an arbitrary function of its arguments. 3
    (c) Evaluate: $L^{-1}\left\{\frac{3p+7}{p^2 - 2p - 3}\right\}$. 3

15. (a) Solve the system of equations 5

$$\frac{dx}{dt} + 4x + 3y = t$$
$$\frac{dy}{dt} + 2x + 5y = e^t$$

    (b) Find the eigen-values and eigen-functions of 5

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{\lambda}{x}y = 0;\ y'(1) = 0 = y'(e^{2\pi}) \text{ where } \lambda > 0 .$$

16. (a) Find the complete integral of the partial differential equation $xpq + yq^2 = 1,\ p \equiv \frac{\partial z}{\partial x},\ q \equiv \frac{\partial z}{\partial y}$, by Charpit's method. 5

(b) Solve $(D+2)^2 y = 4e^{-2t},\ y(0) = -1$ and $y'(0) = 4$, by using Laplace transform technique. 5

17. (a) If y_1 and y_2 are two solutions of $\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$, then show that

$$y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = ce^{-\int p_1 dx}$$

Where c is a constant. What can you say about y_1 and y_2 if $c = 0$?. 3

(b) Evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt$ by Laplace transform technique. 3

(c) Solve the equation: $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ in series, near the ordinary point $x = 0$. 4

18. (a) Use convolution theorem to show that $\mathrm{L}^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\} = \frac{1}{16}\left(e^{2t} - 4te^{-2t} - e^{-2t}\right)$. 4

(b) Solve: $\frac{yzdx}{x^2+y^2} - \frac{xzdy}{x^2+y^2} - \tan^{-1}\frac{y}{x}dz = 0$. 3

(c) Solve the equation: $\left(x^2\frac{d^2}{dx^2} - 3x\frac{d}{dx} + 3\right)y = 2x^3 - x^2$, by factorisation of operators. 3

<u>Unit-II</u>

Answer **<u>any four</u>** questions: 5 x 4

19. (a) Show that the tangent to the curve $x^3 + y^3 = 3axy$, 'a' being a constant at a point$[\neq (0,0)]$ where it meets the parabola $y^2 = ax$ is parallel to the y-axis. 3

(b) Show that the pedal equation of the spiral $r = \operatorname{sech} n\theta$ with respect to pole is $\frac{1}{p^2} = \frac{A}{r^2} + B$, where A and B are constants to be determined by you (n being a constant). 2

20. If the polar equation of a curve is $r = f(\theta)$, where f is an even function of θ, show that its curvature at $\theta = 0$ is $\frac{f(0) - f''(0)}{\{f(0)\}^2}$. 5

21. Show that the points common to the curve

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$$

and its asymtotes lie on the straight line $8x + 2y + 1 = 0$. 5

22. Find the nature and position of the multiple points on the curve $y(y-6) = x^2(x-2)^3 - 9$. 5

23. Find the range of the values of x for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upwards or downwards.
Find its point(s) of inflexion, if any. 3+2

24. (a) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about line $y = b$. Find the volume of the solid thus generated by Pappus theorem. 2

(b) Find the centroid of the arc of the circle $x = a\cos\theta,\ y = a\sin\theta$ which subtends an angle 2α at its centre. 3